A Practical Application of Euler's Method in Biology

The Question

A leaking bucket *initially* contains 2 litres of pure water. Plant food in a solution with a concentration of 1 gram per litre *enters* the solution at a rate of 1 litre per hour. At the same time, the bucket is *leaking* its contents at a rate of 2 litres per hour. For a certain plant, exposure to 0.45 grams of plant food is unsafe. At what times will the contents of the bucket be unsafe for the plant?

Please attempt this problem and feel free to use the help of the Euler's method program found on the website. Detailed solutions are found on the next page.

Detailed Solution

This is similar to a mixing problem we have done in class before. But there is a twist: the volume of liquid in the bucket is not constant.

Since we know that each hour, 1 litre of plant food solution is poured in while 2 litres are drained, we can write an expression for the volume of liquid v, in litres, at any given time $t = x$ minutes. So the volume v at x minutes is

$$v = 2 - x$$

Now that we have an expression for the volume, we can proceed to write a differential for the amount of plant food y, in grams. Recall that

$$\frac{dy}{dx} = \text{rate in} - \text{rate out}$$

Since each hour, 1 litre of solution with a plant food concentration of 1 gram per litre is poured in, the $\text{rate in} = 1 \times 1 = 1$ gram.

Using logic, the rate out is modelled by

$$\text{rate out} = \text{drainage volume} \times \text{drainage concentration}$$

The drainage concentration is the current concentration of plant food in the bucket, which is equals current plant food amount y over current volume v. Hence

$$\text{drainage concentration} = \text{current concentration} = \frac{y}{v}$$

Since we already know that each hour, 2 litres of liquid is drained, the $\text{drainage volume} = 2$. Now plug in the values into the rate out expression and we get

$$\text{rate out} = \text{drainage volume} \times \text{drainage concentration} = 2 \times \frac{y}{v}$$

But recall that we also have $v = 2 - x$ as the expression of the volume liquid any given time $t = x$ minutes. Hence substitute this back into the rate out equation and we have

$$\text{rate out} = 2 \times \frac{y}{v} = \frac{2y}{2 - x}$$

Now that we have both the rate in and the rate out, we can plug them into the differential

$$\frac{dy}{dx} = \text{rate in} - \text{rate out}$$

To get

$$\frac{dy}{dx} = 1 - \frac{2y}{2 - x}$$

Notice how this equation is not a separable differential equation. Hence, to solve it, we can use the computer program.

Since we know that initially, the bucket contains pure water, which means that when time $x = 0$, the amount of plant food $y = 0$. Hence the initial point will be $(0, 0)$.

For the purposes of this question, a step size of 0.1 will suffice.

Next, we proceed to enter the differential

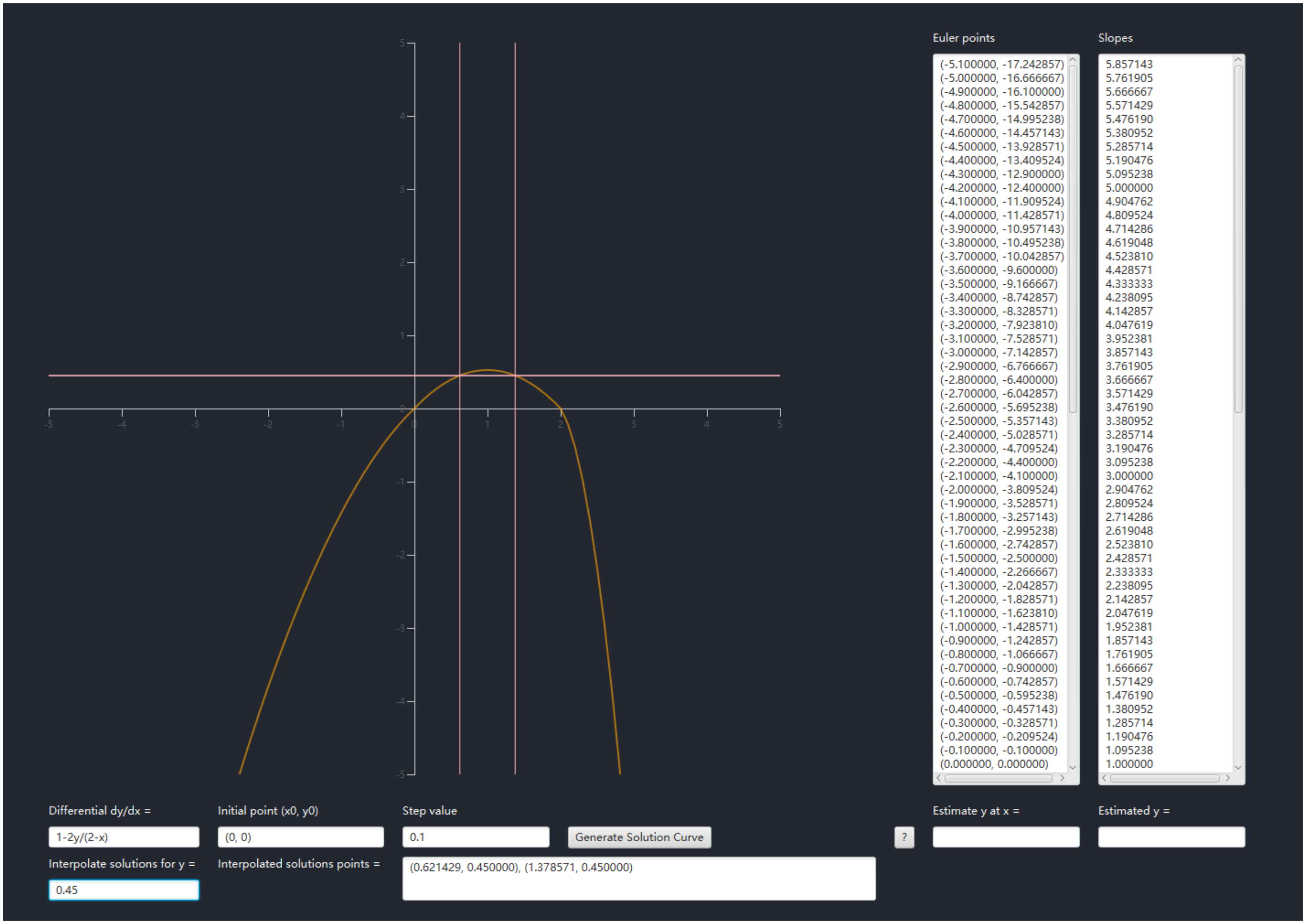
$$\frac{dy}{dx} = 1 - \frac{2y}{2 - x}$$

The initial point $(0, 0)$ and the step size of 0.1 into the respective text fields of the program.

And the generated solution curve is seen on the screen.

Since the question asks when will the amount of plant food be unsafe for the plant, and it also states that it is unsafe for the plant if the amount of plant food reaches 0.45 grams or more, we want to find the ranges of time x for which the amount of plant food $y \geq 0.45$.

Visually, we can tell that the solution curve looks like a downward curving parabola, so there are 2 solutions of x for which $y = 0.45$. We then find the solutions by using the "Interpolate solutions for y" feature of the program by entering 0.45 as the y value.



After hitting "Enter," the program states that solution curve reaches 0.45 at $x \approx 0.621429$ and $x \approx 1.378571$ hours. By looking at the region surrounded by the orange solution curve and the pink crosshairs, it is easy to tell that y, the amount of plant food in grams, satisfies $y \geq 0.45$ for $x \in [0.621429, 1.378571]$ hours.

Therefore, the plant will be unsafe at $0.621429 \sim 1.378571$ hours, which maps to a danger time segment of 37 minutes $\sim$ 1 hour 22 minutes